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A Semi-Analytical Approach for the Approximate Solution of Casting-Mould Heterogeneous System

Muhammad Nadeem*

Faculty of Science, Yibin University, Yibin 644000, China; nadeem@yibinu.edu.cn; jhfang@163.com.

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Abstract

In this work, we study the Variational Iteration Method (VIM) accompanied by He's polynomials for obtaining the solution of the temperature distribution in the casting-mould heterogeneous system with fractional order. The combined form of VIM and the Homotopy Perturbation Method (HPM) is called the Variational Iteration Homotopy Perturbation Method (VIHPM) which yields to implement the system of equations directly. The identification of the Lagrange multiplier is essentially more reliable and higher accurate for such type of problems. The approximate solution converges rapidly to the exact solution which confirms the accuracy of this approach. Some graphical representations are demonstrated to show the validity of this approach.

Keywords: Variational iteration method, Homotopy perturbation method, Casting-mould system, Approximate solution.

1 | Introduction

Partial differential equations play an important and expanded significance in describing a variety of physical phenomena in natural phenomena, such as solid-state physics, fluid mechanics, plasma physics, population dynamics, chemical kinetics, nonlinear optics, protein chemistry, and so on. For appropriate topics, these nonlinear models, as well as their analytic solutions, are quite interesting.

He [1], [2] developed semi-analytical methods such as the Variational Iteration Method (VIM) and the Homotopy Perturbation Method (HPM) for the first time to obtain the approximate solution to linear and nonlinear problems. Later, many scientists confirmed that HPM is a very powerful technique as Edalatpanah and Rashidi [3] confirmed the validity of HPM for finding the approximate solution of the convection-diffusion equation. Yıldırım and Öziş [4] studied HPM for the approximate solution of the Lane-Emden type problem. Domairry and Aziz [5] used HPM to obtain the approximate solution of MHD squeeze flow between two parallel disks. We recommend the researchers study the efficiency of HPM in [6].

Corresponding Author: nadeem@yibinu.edu.cn

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Grzymkowski et al. [7], [8] employed HPM whereas Tripathi and Mishra [9] adopted HPM together with the Laplace transform to determine the temperature distribution in the casting-mould heterogeneous system as a continuous function, which is particularly useful for analyzing the mould. Vanani et al. [10] used a weighted approach based on HPM to solve the heat equation in the cast-mould heterogeneous domain. Later, this proposed approach has also been examined in more than one spatial dimension, indicating that this method has a broader application in nonlinear PDE systems [11], [12]. This study is particularly powerful for fractal theory and fractal calculus, and it can be seen as dependable in getting analytical solutions and suitable for other nonlinear issues [13]-[15].

This study presents the idea of Variational Iteration Homotopy Perturbation Method (VIHPM) to obtain the solution of casting-mould heterogeneous system. The quality of the current method is appropriate to provide the analytical results to the given examples. This study is summarized as follows: In Section 2, we construct the description of the casting-mould system. In Section 3 and 4, we examine the idea of VIM and HPM. We formulate the idea of VIHPM for the implementation of casting-mould system in Section 5. In Section 6, we perform this scheme to a numerical problem to show its capability and efficiency whereas the results and discussion along conclusion are discussed in Section 7 and Section 8.

2 | The Description of the Problem

In this frame, we develop a casting-mould system to examine temperature distribution. Consider, $\Theta(\varsigma,\xi)$ for casting and $\Xi(\varsigma,\xi)$ for mould be two regions on the boundary of the problem [16].

$$\Theta = (\varsigma,\xi){:}\,\zeta \in [\varsigma_1, 0],\ \xi \in [0,\xi^*). \tag{1}$$

$$\Xi = (\varsigma,\xi){:}\,\varsigma \in [0,\varsigma_2],\ \xi \in [0,\xi^*). \tag{2}$$

These two functions $\Theta(\zeta,\xi)$ and $\Xi(\zeta,\xi)$ satisfy the heat transfer equation inside the domains such as

$$\begin{gathered}\frac{\partial\Theta(\varsigma,\xi)}{\partial\xi} = a_1\frac{\partial^2\Theta(\varsigma,\xi)}{\partial\varsigma^2},\ (\varsigma,\xi)\in\Theta,\\ \frac{\partial\Xi(\varsigma,\xi)}{\partial\xi} = a_2\frac{\partial^2\Xi(\varsigma,\xi)}{\partial\varsigma^2},\ (\varsigma,\xi)\in\Xi.\end{gathered} \tag{3}$$

Where a_1 and a_2 are the thermal diffusivity, Θ and Ξ represent the temperature, ς and ξ show the spatial time respectively with the following conditions

$$\begin{aligned}\Theta(\varsigma,0) &= \Xi_1(\varsigma),\text{ on }\Gamma_1\\ \Xi(\varsigma,0) &= \Xi_2(\varsigma),\text{ on }\Gamma_4\\ \Theta(\varsigma_1,\xi) &= \Theta(\xi),\text{ on }\Gamma_3\\ \frac{\Xi(\varsigma_2,\xi)}{\partial\zeta} &= q(\xi),\text{ on }\Gamma_5\\ \Theta(0,\xi) &= \Xi(0,\xi),\text{ on }\Gamma_2\\ \delta_1\frac{\Theta(0,\xi)}{\partial\zeta} &= \delta_2\frac{\Xi(0,\xi)}{\partial\zeta},\text{ on }\Gamma_2\end{aligned} \tag{4}$$

The determination of the approximate solution of the casting-mould system requires careful selection of these boundary parameters.

3 | Variational Iteration Method (VIM)

The VIM was first proposed by He [1], [16] and Nadeem and Li [17]. This method is now widely used in many fields such as Physics [18], Chemistry [19], Biomedical [20] and engineering sciences to study linear and non-linear partial differential equations [21]. Consider a differential problem such as

$$L\Theta(\varsigma,\xi) + N\Theta(\varsigma,\xi) = g(\varsigma,\xi). \tag{5}$$

Where $L = \frac{\partial^k}{\partial t^m}$ is linear and N be a nonlinear operator and $g(\varsigma,\xi)$ represents the source term. The VIM examines the correction functional as follows,

$$\Theta_{n+1}(\varsigma,\xi)=\Theta_n(\varsigma,\xi)+\int_0^{\xi}\lambda(s)\big[L\Theta_n(\zeta,s)+N\widetilde{\Theta}_n(\varsigma,s)-g(\varsigma,s)\big]ds. \tag{6}$$

Eq. (6) is called a correction functional, Θ_0 is called initial approximation, λ denotes Lagrange multiplier is obtained using variational theory and $\widetilde{\Theta}_n$ is called restricted variable such that $\delta\Theta_n = 0$, so

$$\lambda=-1, \text{ for } k=1,$$
$$\lambda=s-\xi, \text{ for } k=2,$$

And in general, for $m \geq 1$

$$\lambda=\frac{(-1)^k(s-\xi)^{k-1}}{(k-1)!}. \tag{7}$$

Eqs. (6) and *(7)* develop the following iteration formula,

$$\Theta_{n+1}(\varsigma,\xi)=\Theta_n(\varsigma,\xi)+\int_0^{\xi}\frac{(-1)^m(s-\xi)^{k-1}}{(k-1)!}\big[Lu_n(\varsigma,s)+N\widetilde{\Theta}_n(\varsigma,s)-g(\varsigma,s)\big]ds. \tag{8}$$

Thus, *Eq. (8)* is said to be a correction functional formula. The values of the Lagrange multiplier λ will be identified optimally via integration by parts. The consecutive approximation $\Theta_{n+1}, n \geq 0$ of the solution Θ can acquire with the help of the Lagrange multiplier. Ultimately, the solution is given by $\Theta = lim_{n\to\infty}\Theta_{n^*}$.

4 | The Basic Idea of the HPM

Let us consider the following non-linear differential problem [22], [23]:

$$T(\Theta)-g(w)=0\,. \tag{9}$$

With boundary conditions

$$\tau\left(\Theta,\frac{\partial\Theta}{\partial n}\right)=0. \tag{10}$$

Where T is a functional and τ is a boundary operator, $g(w)$ is a known function. We separate T into two operators, such as T_1 is a linear and T_2 represents a nonlinear operator respectively. Thus, *Eq. (9)* becomes as

$$L(\Theta)+N(\Theta)-g(w)=0. \tag{11}$$

Let's consider a homotopy $\Xi(r,p)\colon \Omega\times[0,1]\to\mathbb{R}$ that satisfies

$$H(\Xi,p)=(1-p)[L(\Xi)-L(\Theta_0)]+p[L(\Xi)-N(\Xi)-g(w)]. \tag{12}$$

Or

$$H(\Xi,p)=L(\Xi)-L(\Theta_0)+pL(\Theta_0)+p[N(\Xi)-g(w)]=0. \tag{13}$$

Here $p\in[0,1]$, is an embedding parameter, and Θ_0 is an initial guess of *Eq. (9)*, which satisfies the boundary conditions. On the other hand, HPM considered p as a small parameter, and suppose that the solution of *Eq. (13)* can be termed as a power series

$$\Xi=\Xi_0+pv_1+p^2\Xi_2+\cdots. \tag{14}$$

Put *Eq. (14)* into *Eq. (13)* and comparing the identical powers of p, we can find a series of equations in such a form

$$\begin{aligned} p^0\colon \Xi_0-g(\varsigma) &= 0\\ p^1\colon \Xi_1-H(\Xi_0) &= 0\\ p^2\colon \Xi_2-H(\Xi_0,\Xi_1) &= 0^{\cdot}\\ p^3\colon \Xi_3-H(\Xi_0,\Xi_1,\Xi_2) &= 0\end{aligned} \tag{15}$$

Where $H(\Xi_0,\Xi_1,\Xi_2,\cdots,\Xi_j)$ depend upon $\Xi_0,\Xi_1,\Xi_2,\cdots,\Xi_j$ and subjected to He's polynomials. We may calculate them by using the formula

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$$H\left(\Xi_0, \Xi_1, \Xi_2, \cdots, \Xi_j\right) = \frac{1}{j!}\frac{\partial j}{\partial p^j} N\left(\sum_{i=0}^{j} \Xi_i p^i\right)\Big|_{p=0}. \tag{16}$$

Let $p = 1$, the analytical solution of *Eq. (9)* is as follows

$$\Theta = \lim_{p\to 1} \Xi = \Xi_0 + \Xi_1 + \Xi_2 + \Xi_3 + \cdots. \tag{17}$$

It is obvious that the system of nonlinear equations in *Eq. (15)* is easy to solve, and the components $\Xi_i, i \geq 0$ of the HPM can be completely determined, and the series solutions are thus entirely determined.

5 | VIM with He's Polynomials (VIHPM)

In this segment, we construct the strategy of the VIMHP. Therefore, consider the following equation

$$\begin{aligned} L_1\Theta(\varsigma,\xi) + N_1\Theta(\varsigma,\xi) &= g_1(\varsigma,\xi) \\ L_2\Xi(\varsigma,\xi) + N_2\Xi(\varsigma,\xi) &= g_2(\varsigma,\xi) \end{aligned}. \tag{18}$$

According to VIM, we can write the correction functional as follow

$$\begin{aligned} \Theta_{n+1}(\varsigma,\xi) &= \Theta_n(\varsigma,\xi) + \int_0^{\xi} \lambda_1(s)[L_1 u_n(\varsigma,s) + N_1\Theta_n(\varsigma,s) - g_1(\varsigma,s)]ds \\ \Xi_{n+1}(\varsigma,\xi) &= \Xi_n(\varsigma,\xi) + \int_0^{\xi} \lambda_2(s)[L_1\Xi_n(\varsigma,s) + N_2\Xi_n(\varsigma,s) - g_2(\varsigma,s)]ds \end{aligned}. \tag{19}$$

Here λ_1 and λ_2 are Lagrange multipliers. Now, using HPM [24], [25], a system of *Eq. (19)* becomes as

$$\begin{aligned} \sum_{n=0}^{\infty} p^n\Theta_n(\varsigma,\xi) &= \Theta_0(\varsigma,\xi) + p\int_0^{\xi} \lambda_1(s)[N_1(\sum_{n=0}^{\infty} p^n\Theta_n(\varsigma,s)) - g_1(\varsigma,s)]ds \\ \sum_{n=0}^{\infty} p^n\Theta_n(\zeta,\xi) &= \Xi_0(\varsigma,\xi) + p\int_0^{\xi} \lambda_2(s)[N_2(\sum_{n=0}^{\infty} p^n\Xi_n(\varsigma,s)) - g_2(\varsigma,s)]ds \end{aligned}. \tag{20}$$

Equating the components of *Eq. (20)* with similar powers of p, and using p approaches to 1, we can obtain following the formula.

$$\Theta_0(\varsigma,\xi) = \Theta_0(\varsigma,\xi) + \Theta_1(\varsigma,\xi) + \Theta_2(\varsigma,\xi) + \cdots = \lim_{p\to 1}\sum_{n=1}^{\infty} p^n\Theta_n(\varsigma,\xi). \tag{21}$$

$$\Xi(\varsigma,\xi) = \Xi_0(\varsigma,\xi) + \Theta_1(\varsigma,\xi) + \Xi_2(\varsigma,\xi) + \cdots = \lim_{p\to 1}\sum_{\varepsilon=0}^{n} p^n\Xi_n(\varsigma,\xi). \tag{22}$$

6 | Numerical Applications

In this segment, we implement the idea of VIHPM for the approximate solution of casting-mould heterogeneous system. The significant findings demonstrate the validity and consistency of this scheme.

6.1 | Test Problem I

Considering $a_1 = \frac{1}{4}$, $a_2 = 1$, $\delta_1 = 1$, $\delta_2 = 2$, we get the system of *Eq. (3)* such as,

$$\frac{\partial\Theta}{\partial\xi} = \frac{1}{4}\frac{\partial^2\Theta}{\partial\varsigma^2}, \quad \frac{\partial\Xi}{\partial\xi} = \frac{\partial^2\Xi}{\partial\varsigma^2}. \tag{23}$$

With initial condition

$$\Theta(\varsigma,0) = e^{2\varsigma}, \quad \Xi(\varsigma,0) = e^{\varsigma}. \tag{24}$$

According to the strategy, the correctional functional for the system of *Eq. (23)* becomes as

$$\Theta_{n+1}(\varsigma,\xi)=\Theta_n(\varsigma,\xi)+\int_0^{\xi}\lambda_1(s)\left[\frac{\partial\Theta_n}{\partial s}-\frac{1}{4}\frac{\partial^2\Theta_n}{\partial\varsigma^2}\right]ds.$$
$$\Xi_{n+1}(\varsigma,\xi)=\Xi_n(\varsigma,\xi)+\int_0^{\xi}\lambda_2(s)\left[\frac{\partial\Xi_n}{\partial s}-\frac{\partial^2\Xi_n}{\partial\xi^2}\right]ds. \tag{25}$$

Which implies the stationary terms

$$1+\lambda_1(s)=0,\quad \lambda_1'(s=\xi)=0,$$
$$1+\lambda_2(s)=0,\quad \lambda_2(s=\xi)=0.$$

The Lagrange multipliers yields

$$\lambda_a(s)=\lambda_2(s)=-1.$$

Put these values into the system of *Eq. (25)*,

$$\Theta_{n+1}(\zeta,\xi)=\Theta_n(\zeta,\xi)-\int_0^{\xi}\left[\frac{\partial\Theta_n}{\partial s}-\frac{1}{4}\frac{\partial^2\Theta_n}{\partial\zeta^2}\right]ds,$$
$$\Xi_{n+1}(\zeta,\xi)=\Xi_n(\zeta,\xi)-\int_0^{\xi}\left[\frac{\partial\Theta_n}{\partial s}-\frac{\partial^2\Xi_n}{\partial\zeta^2}\right]ds. \tag{26}$$

Applying the VIHPM on the system of *Eq. (26)*, we get

$$\sum_{n=0}^{\infty}p^n\Theta_n(\zeta,\xi)=\Theta_0(\zeta,\xi)+p\int_0^{\xi}\left[\frac{1}{4}\sum_{n=0}^{\infty}p^n\frac{\partial^2\Theta_n}{\partial\zeta^2}\right]ds,$$
$$\sum_{n=0}^{\infty}p^n\Xi_n(\zeta,\xi)=\Xi_0(\zeta,\xi)+p\int_0^{\xi}\left[\sum_{n=0}^{\infty}p^n\frac{\partial^2\Xi_n}{\partial\varsigma^2}\right]ds.$$

Now from *Eq. (23)*, we can select $\Theta_0(\zeta,\xi)=e^{2\zeta},\Xi_0(\zeta,\xi)=e^{\zeta}$. Accordingly, the coefficients of p are, For zero-th order of p, we get

$$p^0:\Theta_0(\varsigma,\xi)=e^{2\zeta}$$
$$p^0:\Xi_0(\zeta,\xi)=e^{\zeta}.$$

For the 1st order of p, we get

$$p^1:\Theta_1(\zeta,\xi)=\int_0^{\xi}\left[\frac{1}{4}\frac{\partial^2\Theta_0}{\partial\zeta^2}\right]ds=e^{2\zeta}\xi$$
$$p^1:\Xi_1(\zeta,\xi)=\int_0^{\xi}\left[\frac{\partial^2\Xi_0}{\partial\varsigma^2}\right]ds=e^{\zeta}\xi.$$

For the 2nd order of p, we get

$$p^2:\Theta_2(\varsigma,\xi)=\int_0^{\xi}\left[\frac{1}{4}\frac{\partial^2\Theta_1}{\partial\varsigma^2}\right]ds=\frac{1}{2}e^{2\zeta}\xi^2$$
$$p^2:\Xi_2(\varsigma,\xi)=\int_0^{\xi}\left[\frac{\partial^2\Xi_1}{\partial\varsigma^2}\right]ds=\frac{1}{2}e^{\zeta}\xi^2.$$

For the 3rd order of p, we get

$$p^3:\Theta_3(\zeta,\xi)=\int_0^{\xi}\left[\frac{1}{4}\frac{\partial^2\Theta_2}{\partial\varsigma^2}\right]ds=\frac{1}{6}e^{2\varsigma}\xi^3$$
$$p^3:\Xi_3(\varsigma,\xi)=\int_0^{\xi}\left[\frac{\partial^2\Xi_2}{\partial\varsigma^2}\right]ds=\frac{1}{6}e^{\zeta}\xi^3.$$

For the 4th order of p, we get

$$\begin{aligned}
&p^4: \Theta_4(\varsigma, \xi) = \int_0^\xi \left[\frac{1}{4}\frac{\partial^2 \Theta_3}{\partial \varsigma^2}\right] ds = \frac{1}{24} e^{2\varsigma} \xi^4 \\
&p^4: \Xi_4(\varsigma, \xi) = \int_0^\xi \left[\frac{\partial^2 \Xi_3}{\partial \varsigma^2}\right] ds = \frac{1}{24} e^{\varsigma} \xi^4 \\
&\vdots
\end{aligned}.$$

The series solutions are therefore given by

$$\Theta(\varsigma, \xi) = \Theta_0 + \Theta_1 + \Theta_2 + \Theta_3 + \cdots = e^{2\zeta}\left(1 + \xi + \frac{\xi^2}{2!} + \frac{\xi^3}{3!} + \frac{\xi^4}{4!} + \cdots\right),$$

$$\Xi(\varsigma, \xi) = \Xi_0 + \Xi_1 + \Xi_2 + \Xi_3 + \cdots = e^{\zeta}\left(1 + \xi + \frac{\xi^2}{2!} + \frac{\xi^3}{3!} + \frac{\xi^4}{4!} + \cdots\right).$$

And hence the exact solution becomes

$$\Theta(\zeta, \xi) = e^{2\zeta+\xi}, \ \Xi(\zeta, \xi) = e^{\zeta+\xi}.$$

6.2 | Test Problem II

Next, considering the system of *Eq. (22)* with the following initial conditions

$$\Theta(\varsigma, 0) = 2 + e^{2\varsigma}, \ \Xi(\varsigma, 0) = e^{\varsigma}. \tag{27}$$

The identification of Lagrange multipliers will be the same and thus according to VIHPM strategy, the coefficients of p are, For zero-th order of p, we get

$$\begin{aligned}
&p^0: \Theta_0(\zeta, \xi) = 2 + e^{2\zeta}, \\
&p^0: \Xi_0(\zeta, \xi) = e^{\zeta}.
\end{aligned} \tag{28}$$

For the 1st order of p, we get

$$\begin{aligned}
&p^1: \Theta_1(\zeta, \xi) = \int_0^\xi \left[\frac{1}{4}\frac{\partial^2 \Theta_0}{\partial \zeta^2}\right] ds = 2 + e^{2\zeta} \xi \\
&p^1: \Xi_1(\zeta, \xi) = \int_0^\xi \left[\frac{\partial^2 \Xi_0}{\partial \zeta^2}\right] ds = e^{\zeta} \xi
\end{aligned}.$$

For the 2nd order of p, we get

$$\begin{aligned}
&p^2: \Theta_2(\zeta, \xi) = \int_0^\xi \left[\frac{1}{4}\frac{\partial^2 \Theta_1}{\partial \zeta^2}\right] ds = \frac{1}{2} e^{2\zeta} \xi^2 \\
&p^2: \Xi_2(\zeta, \xi) = \int_0^\xi \left[\frac{\partial^2 \Xi_1}{\partial \zeta^2}\right] ds = \frac{1}{2} e^{\zeta} \xi^2
\end{aligned}.$$

For the 3rd order of p, we get

$$\begin{aligned}
&p^3: \Theta_3(\varsigma, \xi) = \int_0^\xi \left[\frac{1}{4}\frac{\partial^2 \Theta_2}{\partial \varsigma^2}\right] ds = \frac{1}{6} e^{2\varsigma} \xi^3 \\
&p^3: \Xi_3(\varsigma, \xi) = \int_0^\xi \left[\frac{\partial^2 \Xi_2}{\partial \varsigma^2}\right] ds = \frac{1}{6} e^{\varsigma} \xi^3
\end{aligned}.$$

For the 4th order of p, we get

$$p^4 : \Theta_4(\zeta,\xi) = \int_0^{\xi}\left[\frac{1}{4}\frac{\partial^2\Theta_3}{\partial\varsigma^2}\right]ds == \frac{1}{24}e^{2\zeta}\xi^4$$
$$p^4 : \Xi_4(\zeta,\xi) = \int_0^{\xi}\left[\frac{\partial^2\Xi_3}{\partial\varsigma^2}\right]ds = \frac{1}{24}e^{\zeta}\xi^4 .$$
$$\vdots$$

The series solutions are therefore given by

$$\Theta(\varsigma,\xi) = \Theta_0 + \Theta_1 + \Theta_2 + \Theta_3 + \cdots = 2 + e^{2\zeta}\left(1 + \xi + \frac{\xi^2}{2!} + \frac{\xi^3}{3!} + \frac{\xi^4}{4!} + \cdots\right)$$
$$\Xi(\varsigma,\xi) = \Xi_0 + \Xi_1 + \Xi_2 + \Xi_3 + \cdots = e^{\varsigma}\left(1 + \xi + \frac{\xi^2}{2!} + \frac{\xi^3}{3!} + \frac{\xi^4}{4!} + \cdots\right).$$

And hence the exact solution becomes

$$\Theta(\varsigma,\xi) = 2 + e^{2\zeta+\xi}, \quad \Xi(\varsigma,\xi) = e^{\zeta+\xi}.$$

7 | Results and Discussion

In this segment, we discuss the results of casing-mould system obtained by VIHPM. We see that the identification of the Lagrange multiplier in both examples are very easy to find. We investigate the solution results in the form of the series which converges rapidly to the exact solution in both examples. We observe that the change in the initial condition of casting system does not affect to the mould system which show the validity of VIHPM.

8 | Conclusion

In this work, we implement a semi-analytical approach VIHPM to find the approximate solution of casting-mould heterogeneous system together with intimal conditions. In this work, first, we utilize VIM to construct the correct function for the identification of the Lagrange multiplier and then HPM using He's polynomials has been introduced to get the iterative formula. This strategy is very helpful for finding the series solution of the linear system. It is important to note that the VIHPM has a high rate of convergence towards the exact solutions. Eventually, we observed that the suggested approach is very strong, competent, and convenient in obtaining analytical solutions. We recommend this strategy for several other nonlinear cases, particularly fractal calculus and fractional calculus in science and engineering.

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Conflicts of Interest

The authors declare that they have no conflicts of interest.

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